Announcements

1) New webwork up later today, due the week after spring break.

1) $\det(AB) = \det(A)\det(B)$

Take Math 413 or 452

2) $\det(A^{-1}) = \dfrac{1}{\det(A)}$

If A is in $M_n(\mathbb{R})$ and invertible, $\det(A) \neq 0$ and

$$1 = \det(I_n) = \det(A \cdot A^{-1})$$

$$= \det(A) \cdot \det(A^{-1})$$

(property 1)

Then dividing by $\det(A)$,

$$\det(A^{-1}) = \frac{1}{\det(A)} .$$

3) $\det(A^t) = \det(A)$

$A^t = (c_{i,j})_{i,j=1}^{n}$ where

$c_{i,j} = a_{j,i}$. Then

$$\det(A^t) = \sum_{\substack{\sigma \\ \text{permutation}}} (-1)^{\text{sign}(\sigma)} c_{1,\sigma(1)} \cdot \dots \cdot c_{n,\sigma(n)}$$

$$= \sum_{\substack{\sigma \\ \text{permutation}}} (-1)^{\text{sign}(\sigma)} a_{\sigma(1),1} \cdot \dots \cdot a_{\sigma(n),n}$$

Focus on

$$a_{\sigma(1),1} a_{\sigma(2),2} \cdot \ldots \cdot a_{\sigma(n),n}$$

$$= a_{\sigma(1),\sigma^{-1}(\sigma(1))} \cdot a_{\sigma(2),\sigma^{-1}(\sigma(2))} \cdot \ldots$$

$$\cdot a_{\sigma(n),\sigma^{-1}(\sigma(n))}.$$

Since σ is one-to-one,

σ^{-1} exists and undoes σ.

Since $\{1, 2, \ldots, n\}$ is finite, σ one-to-one implies that for every k, $1 \leq k \leq n$, there is one and only one i, $1 \leq i \leq n$, $\sigma(i) = k$.

Rewrite the product

$$a_{\sigma(1), \sigma^{-1}(\sigma(1))} \cdot \ldots \cdot a_{\sigma(n), \sigma^{-1}(\sigma(n))}$$

$$= a_{k_1, \sigma^{-1}(k_1)} \cdot \ldots \cdot a_{k_n, \sigma^{-1}(k_n)}$$

where $k_i = \sigma(i)$.

Since the order of multiplication doesn't matter, we can rewrite as

$$a_{1,\sigma^{-1}(1)} \, a_{2,\sigma^{-1}(2)} \cdots a_{n,\sigma^{-1}(n)}.$$

Now $(-1)^{\text{sign}(\sigma)} = (-1)^{\text{sign}(\sigma^{-1})}$

since $\sigma \cdot \sigma^{-1} = \text{identity}$, which has sign equal to one.

We can now rewrite

$$\det(A^t)$$

$$= \sum_{\sigma \text{ permutation}} (-1)^{\text{sign}(\sigma)} c_{1,\sigma(1)} \cdot \ldots \cdot c_{n,\sigma(n)}$$

$$= \sum_{\sigma \text{ permutation}} (-1)^{\text{sign}(\sigma)} a_{\sigma(1),1} \cdot \ldots \cdot a_{\sigma(n),n}$$

$$= \sum_{\sigma \text{ permutation}} (-1)^{\text{sign}(\sigma)} a_{1,\sigma^{-1}(1)} \cdot \ldots \cdot a_{n,\sigma^{-1}(n)}$$

$$= \sum_{\sigma \text{ permutation}} (-1)^{\text{sign}(\sigma^{-1})} a_{1,\sigma^{-1}(1)} \cdot \ldots \cdot a_{n,\sigma^{-1}(n)}$$

$$= \sum_{\sigma \text{ permutation}} (-1)^{\operatorname{sign}(\sigma)} a_{1,\sigma(1)} \cdot \ldots \cdot a_{n,\sigma(n)}$$

$$= \det(A)$$

by replacing σ with σ^{-1}.

<u>Interchange two rows</u>

Extra Credit 10 points,

due Friday after break.

You must use the permutation definition!

Remarks: These properties tell exactly what happens to the determinant when you row-reduce a matrix (step-by-step).

Computer Graphics Applications

Section 2.7

Given an image, you want to figure out how to move it forwards/backwards, up/down, and rotate.

Rotations

Start in 2-D.

Given $v = \begin{bmatrix} x \\ y \end{bmatrix}$, want to rotate v counterclockwise by an angle θ.

Write v in polar coordinates.

We can write

$$x = r\cos\varphi$$

$$y = r\sin\varphi$$

where $r = \sqrt{x^2+y^2}$

and $\varphi = \arctan\left(\frac{y}{x}\right)$

if $x \neq 0$. If $x = 0$,

$$\varphi = \pm \frac{\pi}{2}.$$

If we rotate

$$v = \begin{bmatrix} r\cos\varphi \\ r\sin\varphi \end{bmatrix}$$

Counterclockwise by θ,

we want to add θ to

φ to get

$$w = \begin{bmatrix} r\cos(\varphi+\theta) \\ r\sin(\varphi+\theta) \end{bmatrix}$$

Remember trig identities:

$$\cos(\varphi+\theta) = \cos\varphi\cos\theta - \sin\varphi\sin\theta$$

$$\sin(\varphi+\theta) = \sin\varphi\cos\theta + \sin\theta\cos\varphi.$$

Then

$$w = \begin{bmatrix} r(\cos\varphi\cos\theta - \sin\varphi\sin\theta) \\ r(\sin\varphi\cos\theta + \sin\theta\cos\varphi) \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{2\times 2} \begin{bmatrix} r\cos\varphi \\ r\sin\varphi \end{bmatrix}$$

The matrix

$$A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

is called the rotation matrix of angle θ.

It rotates a vector in $\mathbb{R}^2$ counterclockwise through angle θ while not changing the length of the vector!

Example 1:

What value of θ

determines

$$A_\theta = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} ?$$

Set this equal to

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Since the entries have to be equal,

$\cos\theta = \frac{1}{2}$, $\sin\theta = \frac{\sqrt{3}}{2}$,

so $\boxed{\theta = \frac{\pi}{3}}$.

Shifts: A generic term for moving up/down or left/right is shift (or translation).

If $v = \begin{bmatrix} x \\ y \end{bmatrix}$, we want to move v either h units horizontally or k units vertically.

Horizontal shift:

$$\begin{bmatrix} x \\ y \end{bmatrix} \text{ to } \begin{bmatrix} x+h \\ y \end{bmatrix}$$

Vertical shift

$$\begin{bmatrix} x \\ y \end{bmatrix} \text{ to } \begin{bmatrix} x \\ y+k \end{bmatrix}.$$

Neither of these is given by a 2×2 matrix acting on v!

Example 2: Let $h=2$,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}.$$

If shifting by h horizontally is given by a matrix A,

then $A\left(2\begin{bmatrix} 5 \\ 6 \end{bmatrix}\right)$

$$= 2A\begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$$A\begin{bmatrix}5\\6\end{bmatrix} = \begin{bmatrix}7\\6\end{bmatrix},$$

$$\text{so } 2A\begin{bmatrix}5\\6\end{bmatrix} = \begin{bmatrix}14\\12\end{bmatrix}.$$

But

$$A\left(2\begin{bmatrix}5\\6\end{bmatrix}\right) = A\begin{bmatrix}10\\12\end{bmatrix}$$

$$= \begin{bmatrix}12\\12\end{bmatrix}$$

and these are not equal!

Fix:

Move up a dimension!